

XXZ SCALAR PRODUCTS AND KP

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ABSTRACT. Using a Jacobi-Trudi-type identity, we show that the scalar product of a general state and a Bethe eigenstate in a finite-length XXZ spin- $\frac{1}{2}$ chain is (a restriction of) a KP τ function. This leads to a correspondence between the eigenstates and points on Sato's Grassmannian. Each of these points is a function of the rapidities of the corresponding eigenstate, the inhomogeneity variables of the spin chain and the crossing parameter.

0. INTRODUCTION

In [1] we observed that the partition function of the six vertex model on a finite-size square lattice with domain wall boundary conditions, Z_N , is (a restriction of) a KP τ function, for all values of the crossing parameter¹.

In this work, we extend the above result as follows. We prove a Jacobi-Trudi-type identity that implies that certain determinants are restrictions of KP τ functions. Next, for a length- M XXZ spin- $\frac{1}{2}$ chain, we consider the scalar product $\langle \{\lambda\} | \{\mu\}_\beta \rangle$, where $\langle \{\lambda\} |$ is a general state, $\{\lambda\}$ are N free variables, $|\{\mu\}_\beta\rangle$ is a Bethe eigenstate, $\{\mu\}_\beta$ are N variables that satisfy the Bethe equations, and $N \leq M$. We use the Jacobi-Trudi-type identity to show that $\langle \{\lambda\} | \{\mu\}_\beta \rangle$ is a restricted KP τ function, where the (infinitely many) KP time variables are functions of the N free variables $\{\lambda\}$. We obtain an expression for $\langle \{\lambda\} | \{\mu\}_\beta \rangle$ as an expectation value of charged free fermions, then ‘peel off’ the time dependencies to obtain a correspondence between each eigenstate $|\{\mu\}_\beta\rangle$ and a point on Sato's Grassmannian represented by the action of exponentials of fermion bilinears. The coefficients of the bilinears are functions of $\{\mu\}_\beta$, the inhomogeneities of the spin chain and the crossing parameter.

In section 1, we recall basic definitions related to symmetric functions and to KP τ functions. From a KP τ function with no constraints on the time variables we obtain a *restricted* KP τ function by setting the (infinitely many) KP time variables to be power sums in N independent variables. In 2, we prove a Jacobi-Trudi-type identity which implies that certain determinants are restricted KP τ functions. In 3, we recall basic facts related to the algebraic Bethe Ansatz approach to the XXZ spin- $\frac{1}{2}$ chain, and particularly Slavnov's determinant expression for the scalar product of a general state and a Bethe eigenstate [4].

In section 4, we show that the Jacobi-Trudi-type identity introduced in section 2 implies that Slavnov's determinant is a restricted KP τ function. The result of [1], that the domain wall partition function Z_N is a restricted KP τ function follows as a special case. In 5, we write Slavnov's determinant as an expectation value of charged free fermions. In 6, we propose a correspondence between the Bethe eigenstates and points on Sato's Grassmannian, and in 7, we include a number of remarks.

2000 *Mathematics Subject Classification*. Primary 82B20, 82B23.

Key words and phrases. XXZ spin chain. KP. Free fermions.

¹We refer to [2] for an introduction to classical integrable models and KP theory, and to [3] for an introduction to quantum integrable models and the algebraic Bethe Ansatz.

1. SYMMETRIC FUNCTIONS AND KP τ FUNCTIONS

1.1. Elementary symmetric functions. Following [5], the *elementary symmetric function* $e_i\{x\}$ in N variables $\{x\}$ is the i -th coefficient in the generating series

$$(1) \quad \sum_{i=0}^{\infty} e_i\{x\} k^i = \prod_{i=1}^N (1 + x_i k)$$

For example, $e_0\{x\} = 1$, $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $e_2(x_1, x_2) = x_1 x_2$. $e_i\{x\} = 0$, for $i < 0$ and for $i > N$.

1.2. Complete symmetric functions. The *complete symmetric function* $h_i\{x\}$ in N variables $\{x\}$ is the i -th coefficient in the generating series

$$(2) \quad \sum_{i=0}^{\infty} h_i\{x\} k^i = \prod_{i=1}^N \frac{1}{1 - x_i k}$$

For example, $h_0\{x\} = 1$, $h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$, and $h_i\{x\} = 0$ for $i < 0$. In the sequel, we will use the identities

$$(3) \quad h_i\{\widehat{x}_m\} + x_m h_{i-1}\{x\} = h_i\{x\}$$

$$(4) \quad h_i\{\widehat{x}_l\} - h_i\{\widehat{x}_m\} = (x_m - x_l) h_{i-1}\{x\}$$

where $\{\widehat{x}_m\}$ is the set of variables $\{x\}$, with the omission of x_m . Equation (3) follows from (2), and (4) is a re-arrangement of (3).

1.3. Schur functions. The *Schur function* $s_\lambda\{x\}$ indexed by a Young diagram $\lambda = [\lambda_1, \dots, \lambda_r]$ with r non-zero-length rows, $r \leq N$, is

$$(5) \quad s_\lambda\{x\} = \frac{\det \left(x_i^{\lambda_j - j + N} \right)_{1 \leq i, j \leq N}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} = \det \left(h_{\lambda_i - i + j}\{x\} \right)_{1 \leq i, j \leq N}$$

where $\lambda_i = 0$, for $r + 1 \leq i \leq N$. For example, $s_\phi\{x\} = 1$, $s_{[1]}(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $s_{[1,1]}(x_1, x_2) = x_1 x_2$. The first and second equalities in Equation (5) are the definition and the *Jacobi-Trudi identity* for the Schur functions, $s_\lambda\{x\}$, which form a basis for the ring of symmetric functions in $\{x\}$.

1.4. Character polynomials. The *one-row character polynomial* $\chi_i\{t\}$ is the i -th coefficient in the generating series

$$(6) \quad \sum_{i=0}^{\infty} \chi_i\{t\} k^i = \exp \left(\sum_{i=1}^{\infty} t_i k^i \right)$$

For example, $\chi_0\{t\} = 1$, $\chi_1\{t\} = t_1$, $\chi_2\{t\} = \frac{t_1^2}{2} + t_2$, $\chi_3\{t\} = \frac{t_1^3}{6} + t_1 t_2 + t_3$, and $\chi_i\{t\} = 0$ for $i < 0$. The *character polynomial* $\chi_\lambda\{t\}$ indexed by a Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$, with r non-zero-length rows, $r \leq N$, is

$$(7) \quad \chi_\lambda\{t\} = \det \left(\chi_{\lambda_i - i + j}\{t\} \right)_{1 \leq i, j \leq n}$$

For example $\chi_{[1,1]}\{t\} = \frac{t_1^2}{2} - t_2$, $\chi_{[2,1]}\{t\} = \frac{t_1^3}{3} - t_3$. Notice that $\chi_\lambda\{t\}$ is a polynomial in infinitely many variables $\{t\}$, in the sense that it can depend on all t_i , for $i \leq |\lambda|$, where $|\lambda|$ is the sum of the lengths of all rows in λ . As $|\lambda| \rightarrow \infty$, more

time variables t_i contribute to $\chi_\lambda\{t\}$, which form a basis for the ring of symmetric functions in the infinitely-many variables $\{t\}$.

1.5. From character polynomials to Schur functions. The restriction $t_m \rightarrow \frac{1}{m} \sum_{i=1}^N x_i^m$ sends $\chi_i\{t\} \rightarrow h_i\{x\}$. Consequently, the character polynomials $\chi_\lambda\{t\}$ in infinitely many time variables $\{t\}$, where λ has r non-zero-length rows, become equal to the Schur functions $s_\lambda\{x\}$ in N variables $\{x\}$, where $N \geq r$, by setting $t_m \rightarrow \frac{1}{m} \sum_{i=1}^N x_i^m$, for $m \geq 1$.

1.6. KP τ functions. A function $\tau\{t\}$ in the infinitely-many time variables $\{t\}$ is a KP τ function if and only if it can be expanded in the basis of character polynomials $\chi_\lambda\{t\}$ as

$$(8) \quad \tau\{t\} = \sum_{\lambda} c_{\lambda} \chi_{\lambda}\{t\}$$

where the coefficients c_{λ} satisfy Plücker relations [2, 6].

1.7. Restricted KP τ functions. We define a *restricted* τ function $\tau\{x\}$ to be a τ function $\tau\{t\}$ whose infinitely many time variables $\{t\}$ have been set to $t_m \rightarrow \frac{1}{m} \sum_{i=1}^N x_i^m$ for $m \geq 1$, that is, power sums in the N variables $\{x\}$. A restricted τ function is symmetric in the N variables $\{x\}$, since from (8) it has the form

$$(9) \quad \tau\{x\} = \sum_{\lambda \subseteq [(M-1)^N]} c_{\lambda} s_{\lambda}\{x\}$$

where the coefficients c_{λ} satisfy Plücker relations, and $M \geq 1$. On the right hand side of Equation (9), we indicated that the sum is over all Young diagrams $\lambda \subseteq [(M-1)^N]$. That the maximal number of allowed rows in λ is N is due to the fact that there are only N independent variables $\{x\}$. That the number of columns is $M-1$, is due to the fact that in this work, we consider only polynomials in $\{x\}$. The precise value of M is at this stage unspecified. In the sequel, $\tau\{t\}$ functions of the infinitely many variables $\{t\}$ are unrestricted, while the $\tau\{x\}$ functions of the N variables $\{x\}$ are restricted.

1.8. Determinants that are restricted KP τ functions. There are determinants that can be put in the form on the right hand side of Equation (9), hence they are restricted KP τ function, as shown in the following lemma.

Lemma 1. Let \mathcal{H} be a matrix with entries $\mathcal{H}_{ij} = h_{j-i}\{x\}$, $1 \leq i \leq N$, $1 \leq j \leq N + M - 1$, where $\{x\}$ is a set of N variables. Let \mathcal{C} be a constant matrix with entries $\mathcal{C}_{ij} = c_{ij}$, $1 \leq i \leq N + M - 1$ and $1 \leq j \leq N$. We choose $M \geq 1$. The product $\mathcal{H}\mathcal{C}$ is an $N \times N$ matrix, and $\det(\mathcal{H}\mathcal{C})$ is a restricted τ function of the KP hierarchy.

Proof. In the following calculation, all determinants are over $N \times N$ matrices, so we do not need to write the ranges of the row and column indices explicitly.

(10)

$$\begin{aligned}
\det(\mathcal{HC}) &= \det \left(\sum_{k=1}^{N+M-1} h_{k-i}\{x\} c_{kj} \right) \\
&= \sum_{1 \leq k_1 < \dots < k_N \leq N+M-1} \det \left(h_{k_j-i}\{x\} \right) \det \left(c_{k_i,j} \right) \\
&= \sum_{0 \leq \lambda_N \leq \dots \leq \lambda_1 \leq M-1} \det \left(h_{\lambda_{(N-j+1)+j-i}}\{x\} \right) \det \left(c_{\lambda_{(N-i+1)+i},j} \right) \\
&= \sum_{0 \leq \lambda_N \leq \dots \leq \lambda_1 \leq M-1} \det \left(h_{\lambda_i-i+j}\{x\} \right) \det \left(c_{\lambda_{(N-i+1)+i},j} \right) \\
&= \sum_{\lambda \subseteq [(M-1)^N]} c_\lambda s_\lambda \{x\}
\end{aligned}$$

where $c_\lambda = \det \left(c_{\lambda_{(N-i+1)+i},j} \right)$. By construction, the coefficients c_λ obey Plücker relations [2, 6], and the required form (9) is obtained.

2. A JACOBI-TRUDI-TYPE IDENTITY

In the following, we show that an expression that shows up frequently in the theory of quantum integrable models can be put in the form of the left hand side of Equation (10), hence it is also a restricted KP τ function.

Lemma 2. Let κ be an $(N+M-1) \times N$ matrix with coefficients κ_{ij} that do not depend on $\{x\}$, and $M \geq 1$. Then

(11)

$$\frac{\det \left(\sum_{k=1}^{N+M-1} x_i^{k-1} \kappa_{kj} \right)_{1 \leq i,j \leq N}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} = \det \left(\sum_{k=1}^{N+M-1} h_{k-i}\{x\} \kappa_{k,N-j+1} \right)_{1 \leq i,j \leq N}$$

Proof. The first step in the proof is to cancel the factor in the denominator of the left hand side of Equation (11). We define the operator \mathcal{R}_i which acts on the i -th row of an arbitrary $N \times N$ matrix \mathcal{A} . In component form, we write

$$(12) \quad \left(\mathcal{R}_i \mathcal{A} \right)_{ij} = \mathcal{A}_{ij} - \delta_{ii'} \mathcal{A}_{i+1,j}$$

Let $\mathcal{M}^{(1)}$ be an $N \times N$ matrix with components $\mathcal{M}_{ij}^{(1)} = \sum_{k=1}^{N+M-1} x_i^{k-1} \kappa_{kj}$. Observing that $\mathcal{M}_{ij}^{(1)} = \sum_{k=1}^{N+M-1} h_{k-1}(x_i) \kappa_{kj}$, and making repeated use of Equation (4), we find the string of relations

$$\begin{aligned}
(13) \quad \det \left(\prod_{i=1}^{N-1} \mathcal{R}_i \mathcal{M}^{(1)} \right) &= \det \mathcal{M}^{(2)} \prod_{i=1}^{N-1} (x_i - x_{i+1}) \\
\det \left(\prod_{i=1}^{N-2} \mathcal{R}_i \mathcal{M}^{(2)} \right) &= \det \mathcal{M}^{(3)} \prod_{i=1}^{N-2} (x_i - x_{i+2}) \\
&\vdots \\
\det \left(\mathcal{R}_1 \mathcal{M}^{(N-1)} \right) &= \det \mathcal{M}^{(N)} (x_1 - x_N)
\end{aligned}$$

where $\prod_{i=1}^n \mathcal{R}_i = \mathcal{R}_n \dots \mathcal{R}_1$. For simplicity, we show only the components of the last determinant

$$(14) \quad \mathcal{M}_{ij}^{(N)} = \sum_{k=1}^{N+M-1} h_{k-N+i-1} \{x\}_i \kappa_{kj}$$

where $\{x\}_i$ is the set of $N-i+1$ variables x_i, \dots, x_N . Due to invariance of the determinant under such row operations, we can combine the string of relations in Equation (13) into a single equation

$$(15) \quad \frac{\det \mathcal{M}^{(1)}}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} = \det \mathcal{M}^{(N)}$$

We have almost succeeded in obtaining the right-hand-side of Equation (11), except that the complete symmetric functions in $\mathcal{M}_{ij}^{(N)}$, as shown in Equation (14) depend on sets of variables of different cardinalities.

The second step in the proof is to use row operations to introduce dependence on the full set of N variables $\{x\}$ in each of the complete symmetric functions. Introduce the operator $\mathcal{R}_i(x)$, which differs from that used in the first step in that it depends on a (single) variable x and acts on the i -th row of an arbitrary matrix \mathcal{A} to give

$$(16) \quad \left(\mathcal{R}_{i'}(x) \mathcal{A} \right)_{ij} = \mathcal{A}_{ij} + \delta_{ii'} x \mathcal{A}_{i-1,j}$$

Using Equation (14) for $\mathcal{M}_{ij}^{(N)}$ and Equation (3) repeatedly, we obtain another string of relations

$$(17) \quad \begin{aligned} \det \left(\prod_{i=N}^2 \mathcal{R}_i(x_{i-1}) \mathcal{M}^{(N)} \right) &= \det \mathcal{M}^{(N+1)} \\ \det \left(\prod_{i=N}^3 \mathcal{R}_i(x_{i-2}) \mathcal{M}^{(N+1)} \right) &= \det \mathcal{M}^{(N+2)} \\ &\vdots \\ \det \left(\mathcal{R}_N(x_1) \mathcal{M}^{(2N-2)} \right) &= \det \mathcal{M}^{(2N-1)} \end{aligned}$$

where again, for simplicity, we show only the components of the final determinant

$$(18) \quad \mathcal{M}_{ij}^{(2N-1)} = \sum_{k=1}^{N+M-1} h_{k-N+i-1} \{x\} \kappa_{kj}$$

and the complete symmetric functions now depend on the full set of N variables $\{x\}$. Again, since determinants are invariant under these row operations, the string of relations in Equation (17) can be combined into the single equation

$$(19) \quad \det \mathcal{M}^{(N)} = \det \mathcal{M}^{(2N-1)} = \det \left(\sum_{k=1}^{N+M-1} h_{k-i} \{x\} \kappa_{k, N-j+1} \right)$$

where we have performed trivial index shifts in $\det \mathcal{M}^{(2N-1)}$ to obtain the right hand side of Equation (19). Recalling Equation (15), the proof is complete. The

classical Jacobi-Trudi identity is recovered from Equation (11) by setting $M - 1 = \lambda_1$, and $\kappa_{kj} = \delta_{k-1, \lambda_j - j + N}$.

2.1. Corollary. Writing $c_{kj} = \kappa_{k, N-j+1}$, it follows that the left hand side of Equation (10) is identical to the right hand side of Equation (11), and that any expression in the form of the left hand side of Equation (11) is a restricted KP τ function.

3. XXZ AND THE ALGEBRAIC BETHE ANSATZ

3.1. The XXZ Hamiltonian. Let $\sigma_i^{x,y,z}$ be Pauli matrices acting in space V_i isomorphic to \mathbb{C}^2 . The Hamiltonian of a length- M XXZ spin- $\frac{1}{2}$ chain is

$$(20) \quad H = \sum_{i=1}^M \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right)$$

where periodicity $\sigma_{M+1}^{x,y,z} = \sigma_1^{x,y,z}$ is imposed. The eigenstates and eigenvalues of H are obtained using the algebraic Bethe Ansatz, as we outline below.

3.2. The R -matrix. Consider the R -matrix

$$(21) \quad R_{ab}(\lambda, \mu) = \begin{pmatrix} [\lambda - \mu + \gamma] & 0 & 0 & 0 \\ 0 & [\lambda - \mu] & [\gamma] & 0 \\ 0 & [\gamma] & [\lambda - \mu] & 0 \\ 0 & 0 & 0 & [\lambda - \mu + \gamma] \end{pmatrix}_{ab}$$

where $[x] = e^x - e^{-x}$. The subscripts of R_{ab} indicate that it acts in the tensor product $V_a \otimes V_b$, where each V_i is isomorphic to \mathbb{C}^2 . R_{ab} satisfies the Yang-Baxter equation in $V_a \otimes V_b \otimes V_c$

$$(22) \quad R_{ab}(\lambda, \mu) R_{ac}(\lambda, \nu) R_{bc}(\mu, \nu) = R_{bc}(\mu, \nu) R_{ac}(\lambda, \nu) R_{ab}(\lambda, \mu)$$

The R -matrix in Equation (21) plays an essential role in the algebraic Bethe Ansatz approach for a variety of quantum integrable models [3], one of which is the XXZ spin- $\frac{1}{2}$ chain.

3.3. The L -operator. To solve a quantum integrable model using the algebraic Bethe Ansatz, it is necessary to specify the L -operator, which is specific to the model under consideration. For the XXZ spin- $\frac{1}{2}$ chain, the L -operator is $L_{ab}(\lambda, \nu) = R_{ab}(\lambda, \nu)$. Using the Yang-Baxter equation (22), the L -operator satisfies the local intertwining relation

$$(23) \quad R_{ab}(\lambda, \mu) L_{ac}(\lambda, \nu) L_{bc}(\mu, \nu) = L_{bc}(\mu, \nu) L_{ac}(\lambda, \nu) R_{ab}(\lambda, \mu)$$

3.4. The monodromy T -matrix. The monodromy matrix T_a is defined by

$$(24) \quad T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a = L_{a1}(\lambda, \nu_1) \dots L_{aM}(\lambda, \nu_M)$$

where it is conventional to suppress dependence on the inhomogeneities ν_i in T_a , and it is implicit that each of the operators A , B , C , and D acts in the tensor product $V_1 \otimes \dots \otimes V_M$. Using Equation (23) inductively, one derives the intertwining relation

$$(25) \quad R_{ab}(\lambda, \mu) T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda) R_{ab}(\lambda, \mu)$$

which contains all the algebraic relations between the operators A , B , C , and D .

3.5. The transfer \mathcal{T} -matrix. The trace of the monodromy matrix $\text{Tr}_a T_a(\lambda) = \mathcal{T} = A(\lambda) + D(\lambda)$, is the *transfer matrix* of the spin chain.

3.6. The Bethe equations. Given a set of N rapidities $\{\mu\}$, the Bethe equations are the set of equations

$$(26) \quad (-)^{N-1} \frac{a(\mu_i)}{d(\mu_i)} \prod_{j \neq i}^N \frac{[\mu_j - \mu_i + \gamma]}{[\mu_i - \mu_j + \gamma]} = 1$$

for $1 \leq i \leq N$, where the functions $a(\mu)$ and $d(\mu)$ are the respective eigenvalues of the $A(\mu)$ and $D(\mu)$ operators

$$(27) \quad A(\mu)|0\rangle = a(\mu)|0\rangle, \quad D(\mu)|0\rangle = d(\mu)|0\rangle$$

The explicit form of $a(\mu)$ and $d(\mu)$ is specific to the model under consideration. In the XXZ spin chain, $a(\mu) = \prod_{i=1}^M [\mu - \nu_i + \gamma]$, $d(\mu) = \prod_{i=1}^M [\mu - \nu_i]$.

The Bethe equations are necessary to show that certain states are eigenvectors of the transfer matrix \mathcal{T} , and the Hamiltonian H , of the spin chain.

3.7. The Bethe eigenstates. The eigenstates of the transfer matrix $\text{Tr}_a T_a(\lambda) = A(\lambda) + D(\lambda)$ are also eigenstates of the Hamiltonian [3]. The problem of finding eigenstates of the XXZ Hamiltonian is that of finding states $|\Psi\rangle$ that satisfy

$$(28) \quad \left(A(\lambda) + D(\lambda) \right) |\Psi\rangle = \kappa_\Psi(\lambda) |\Psi\rangle$$

for some eigenvalue $\kappa_\Psi(\lambda)$. The algebraic Bethe Ansatz for the eigenstates is

$$(29) \quad |\Psi\rangle = |\{\mu\}_\beta\rangle = B(\mu_1) \dots B(\mu_N)|0\rangle$$

where $N \leq M$, the reference state $|0\rangle = \otimes^M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $|\{\mu\}_\beta\rangle$ indicates a state of the form (29), with rapidities that satisfy the Bethe equations (26). We call such a state a *Bethe eigenstate*.

3.8. Scalar products. It is possible to define a space dual to that of Equation (29)

$$(30) \quad \langle \{\lambda\} | = \langle 0 | C(\lambda_1) \dots C(\lambda_N)$$

where $\langle 0 | = \otimes^M (1 \ 0)$, and take scalar products of the two

$$(31) \quad \langle \{\lambda\} | \{\mu\} \rangle = \langle 0 | C(\lambda_1) \dots C(\lambda_N) B(\mu_1) \dots B(\mu_N) | 0 \rangle$$

The scalar product (31) can be written as a complicated sum over a product of two determinants, [3]. Simpler expressions exist when the variables $\{\lambda\}$ and/or $\{\mu\}$ satisfy the Bethe equations.

3.9. Bethe scalar products. Let us consider the case when the set $\{\lambda\}$ are free variables, and the set $\{\mu\}$ satisfy the Bethe equations. We will denote the latter set by $\{\mu\}_\beta$. Scalar products of the form $\langle \{\lambda\} | \{\mu\}_\beta \rangle$ play a central role in the calculation of correlation functions [11].

3.10. Determinant expressions for the Bethe scalar products. Following [4], the Bethe scalar product (31) can be written as

$$(32) \quad \langle \{\lambda\} | \{\mu\}_\beta \rangle = \frac{[\gamma]^N \prod_{i,j=1}^N [\lambda_i - \mu_j + \gamma]}{\prod_{1 \leq i < j \leq N} [\lambda_i - \lambda_j][\mu_j - \mu_i]} \prod_{k=1}^N d(\lambda_k) d(\mu_k) \det \Omega$$

where the components of the $N \times N$ matrix Ω are

$$(33) \quad \Omega_{ij} = \frac{1}{[\lambda_i - \mu_j][\lambda_i - \mu_j + \gamma]} - \frac{(-)^N}{[\mu_j - \lambda_i][\mu_j - \lambda_i + \gamma]} \frac{a(\lambda_i)}{d(\lambda_i)} \prod_{k=1}^N \frac{[\mu_k - \lambda_i + \gamma]}{[\lambda_i - \mu_k + \gamma]}$$

Using the explicit form of the functions $a(\lambda)$ and $d(\lambda)$ for the case of the XXZ model [3], we obtain

$$(34) \quad \langle \{\lambda\} | \{\mu\}_\beta \rangle = \frac{[\gamma]^N \prod_{i,j=1}^N [\lambda_i - \mu_j + \gamma]}{\prod_{1 \leq i < j \leq N} [\lambda_i - \lambda_j][\mu_j - \mu_i]} \prod_{k=1}^N \prod_{l=1}^M [\lambda_k - \nu_l][\mu_k - \nu_l] \det \Omega$$

where the components of Ω now become

$$(35) \quad \Omega_{ij} = \frac{1}{[\lambda_i - \mu_j][\lambda_i - \mu_j + \gamma]} - \frac{(-)^N}{[\mu_j - \lambda_i][\mu_j - \lambda_i + \gamma]} \prod_{k=1}^M \frac{[\lambda_i - \nu_k + \gamma]}{[\lambda_i - \nu_k]} \prod_{l=1}^N \frac{[\mu_l - \lambda_i + \gamma]}{[\lambda_i - \mu_l + \gamma]}$$

We refer to [4] for details of the proof.

4. THE SCALAR PRODUCT IS A RESTRICTED KP τ FUNCTION

We now bring the determinant in Equation (34) to the form of a restricted KP τ function. The first step is to rewrite it so that it is more clearly a trigonometric polynomial in $\{\lambda\}$. This is done by absorbing the products in the numerator of (34) into the determinant, to obtain

$$(36) \quad \langle \{\lambda\} | \{\mu\}_\beta \rangle = \frac{[\gamma]^N \det \Omega'}{\prod_{1 \leq i < j \leq N} [\lambda_i - \lambda_j][\mu_j - \mu_i]}$$

where the components of Ω' are

$$(37) \quad \Omega'_{ij} = \frac{\prod_{k=1}^M [\mu_j - \nu_k]}{[\lambda_i - \mu_j]} \times \left(\prod_{k=1}^M [\lambda_i - \nu_k] \prod_{l \neq j}^N [\lambda_i - \mu_l + \gamma] + (-)^N \prod_{k=1}^M [\lambda_i - \nu_k + \gamma] \prod_{l \neq j}^N [\mu_l - \lambda_i + \gamma] \right)$$

Using the Bethe equations (26) to rewrite the factor $\prod_{k=1}^M [\mu_j - \nu_k]$ in the numerator of the right hand side of Equation (37), and extracting an overall factor from the resulting determinant, we obtain

$$(38) \quad \langle \{\lambda\} | \{\mu\}_\beta \rangle = \frac{[\gamma]^N \det \Omega''}{\left(\prod_{1 \leq i < j \leq N} [\lambda_i - \lambda_j][\mu_j - \mu_i] \right) \prod_{i \neq j} [\mu_i - \mu_j + \gamma]}$$

where the components of the $N \times N$ matrix Ω'' are

$$(39) \quad \Omega''_{ij} = \frac{(-)^N}{[\lambda_i - \mu_j]} \left(\prod_{k=1}^M [\lambda_i - \nu_k + \gamma][\mu_j - \nu_k] \prod_{l \neq j}^N [\mu_l - \lambda_i + \gamma][\mu_j - \mu_l + \gamma] \right. \\ \left. - \prod_{k=1}^M [\lambda_i - \nu_k][\mu_j - \nu_k + \gamma] \prod_{l \neq j}^N [\lambda_i - \mu_l + \gamma][\mu_l - \mu_j + \gamma] \right)$$

Equation (38) is a trigonometric polynomial in the variables $\{\lambda\}$, since all poles in the expression are removable. This can be seen as follows. The poles at $\lambda_i = \lambda_j$, $i \neq j$ are canceled by the zeros in the determinant when two rows become equal. Also, the poles at $\lambda_i = \mu_j$ within Ω''_{ij} are canceled by corresponding zeros in the numerator. We now come to the statement of our result.

Consider the normalized scalar product

$$(40) \quad \left(\prod_{i=1}^N e^{(M-1)(\lambda_i + \mu_i)} \prod_{j=1}^M e^{2N\nu_j} \right) \langle \{\lambda\} | \{\mu\}_\beta \rangle$$

and make the change of variables

$$(41) \quad \{e^{2\lambda_i}, e^{2\mu_i}, e^{2\nu_i}, e^\gamma\} \rightarrow \{x_i, y_i, z_i, q\}$$

to obtain

$$(42) \quad \left(\prod_{i=1}^N e^{(M-1)(\lambda_i + \mu_i)} \prod_{j=1}^M e^{2N\nu_j} \right) \langle \{\lambda\} | \{\mu\}_\beta \rangle \rightarrow \langle \{x\} | \{y\}_\beta \rangle$$

where

$$(43) \quad \langle \{x\} | \{y\}_\beta \rangle = \frac{(q - q^{-1})^N \det \Omega}{\left(\prod_{1 \leq i < j \leq N} (x_i - x_j)(y_j - y_i) \right) \prod_{i \neq j} (y_i q - y_j q^{-1})}$$

and the components of Ω are

$$(44) \quad \Omega_{ij} = \frac{1}{x_i - y_j} \left(\prod_{k=1}^M (x_i - z_k)(y_j q - z_k q^{-1}) \prod_{l \neq j}^N (x_i q - y_l q^{-1})(y_j q^{-1} - y_l q) \right. \\ \left. - \prod_{k=1}^M (x_i q - z_k q^{-1})(y_j - z_k) \prod_{l \neq j}^N (x_i q^{-1} - y_l q)(y_j q - y_l q^{-1}) \right)$$

Lemma 3. Let $\{y\}_\beta$ be a set of N variables that correspond to the set $\{\mu\}_\beta$ that satisfy the Bethe equations (26), then $\langle \{x\} | \{y\}_\beta \rangle$ as defined in Equation (43), is a restricted KP τ function in the variables $\{x\}$, where $x_i = e^{2\lambda_i}$.

Proof. The proof is a corollary of Lemma 1 and Lemma 2. As we are considering the expression in Equation (43) to be a restricted KP τ function in the $\{x\}$ variables, we treat all pre-factors *not* involving $\{x\}$ as constants, and we need only show that $\frac{\det \Omega}{\prod_{1 \leq i < j \leq N} (x_i - x_j)}$ is in the form of a restricted KP τ function. We start by writing

$$(45) \quad \Omega_{ij} = \frac{\sum_{k=1}^{N+M} x_i^{k-1} \rho_{kj}}{x_i - y_j}$$

where the coefficients ρ_{kj} are given by

$$(46) \quad \rho_{kj} = \left(\prod_{m=1}^M (y_j q - z_m q^{-1}) \right) \left(\prod_{n \neq j}^N (y_j - y_n q^2) \cdot e_{(M+N-k)} \{ -\widehat{y}_j q^{-2} \} \{ -z \} \right) \\ - \left(\prod_{m=1}^M (y_j q - z_m q) \right) \left(\prod_{n \neq j}^N (y_j - y_n q^{-2}) \cdot e_{(M+N-k)} \{ -\widehat{y}_j q^2 \} \{ -z q^{-2} \} \right)$$

and $e_k \{ \widehat{y}_j \} \{ z \}$ is the k -th elementary symmetric polynomial (1) in the set of variables $\{y\} \cup \{z\}$ with the omission of y_j . From Equation (44), the numerator of Ω_{ij} vanishes for $x_i = y_j$, hence the right hand side of Equation (45) is a polynomial in $\{x\}$

$$(47) \quad \Omega_{ij} = \sum_{k=1}^{N+M-1} x_i^{k-1} \left(- \sum_{l=1}^k y_j^{l-k-1} \rho_{lj} \right)$$

Combining the results of Equation (45–47), we obtain

$$(48) \quad \Omega_{ij} = \sum_{k=1}^{N+M-1} x_i^{k-1} \kappa_{kj}, \quad \text{where} \quad \kappa_{kj} = - \sum_{l=1}^k y_j^{l-k-1} \rho_{lj}$$

The coefficients κ_{kj} do not depend on $\{x\}$ or the row-index i , and the required result follows from Equation (11).

4.1. General scalar products are not τ functions. We have verified by explicit checks of nontrivial cases that $\langle \{\lambda\} | \{\mu\} \rangle$, where neither sets of rapidities satisfy the Bethe equations, is *not* a restricted KP τ function.

4.2. The domain wall partition function as a special case. The Bethe scalar product $\langle \{\lambda\} | \{\mu\}_\beta \rangle$ contains two sets of rapidities $\{\lambda\}$ and $\{\mu\}_\beta$, each of cardinality N , and a set of inhomogeneities $\{\nu\}$ of cardinality M . In the particular case when $M = N$, the scalar product reduces to a product of two domain wall partition functions [3]

$$(49) \quad \langle \{\lambda\} | \{\mu\}_\beta \rangle = \langle 0 | C(\lambda_1) \dots C(\lambda_N) | 1 \rangle \langle 1 | B(\mu_1) \dots B(\mu_N) | 0 \rangle \\ = Z_N \left(\{\lambda\}, \{\nu\} \right) Z_N \left(\{\mu\}_\beta, \{\nu\} \right)$$

where $|1\rangle = \otimes^N \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\langle 1| = \otimes^N (0 \ 1)$. Multiplying both sides of Equation (49) by $\prod_{i=1}^N e^{(N-1)(\lambda_i + \mu_i)} e^{2N\nu_i}$ and using the change of variables in Equation (41), we obtain

$$(50) \quad \langle \{x\} | \{y\}_\beta \rangle = Z_N \left(\{x\}, \{z\} \right) Z_N \left(\{y\}_\beta, \{z\} \right)$$

From Lemma 3, $\langle \{x\} | \{y\}_\beta \rangle$ on the right hand side of Equation (50) is a restricted KP τ function in the variables $\{x\}$. Considering $Z_N \left(\{y\}_\beta, \{z\} \right)$ to be a multiplicative constant, we conclude that $Z_N \left(\{x\}, \{z\} \right)$ is a restricted KP τ function in the variables $\{x\}$. This is the result obtained in [1].

5. FERMIONIC EXPECTATION VALUES

In this section we reconsider the identity given in Equation (11) and express it as a vacuum expectation value of charged free fermions, with restricted time variables. The derivation of the result proceeds *mutatis mutandis* as that of an analogous result for the domain wall partition function discussed in detail in [1], hence we only give the final results and refer to [1] for the details.

5.1. Charged free fermions and vacuum states. The free fermion operators $\{\psi_n, \psi_n^*\}$, $n \in \mathbb{Z}$, with charges $\{+1, -1\}$ and energies n , satisfy the anti-commutation relations

$$(51) \quad \left. \begin{aligned} [\psi_m, \psi_n]_+ &= 0 \\ [\psi_m^*, \psi_n^*]_+ &= 0 \\ [\psi_m, \psi_n^*]_+ &= \delta_{m,n} \end{aligned} \right\} \quad \forall m, n \in \mathbb{Z}$$

The vacuum states $\langle 0|$ and $|0\rangle$ are defined by the actions

$$(52) \quad \left. \begin{aligned} \langle 0|\psi_n = \psi_m|0\rangle &= 0, \\ \langle 0|\psi_m^* = \psi_n^*|0\rangle &= 0, \end{aligned} \right\} \quad \forall m < 0, n \geq 0$$

and the inner product normalization

$$(53) \quad \langle 0|0\rangle = 1$$

5.2. Creation/annihilation operators and normal ordering. The annihilation operators are those which annihilate the vacuum state $|0\rangle$, that is $\psi_m|0\rangle = 0$, $m < 0$, and $\psi_n^*|0\rangle = 0$, $n \geq 0$, while all other operators, $\psi_m|0\rangle \neq 0$, $m \geq 0$, and $\psi_n^*|0\rangle \neq 0$, $n < 0$ are creation operators. The normal-ordered product is defined, as usual, by placing annihilation operators to the right of creation operators

$$(54) \quad : \psi_i \psi_j^* : = \psi_i \psi_j^* - \langle 0|\psi_i \psi_j^*|0\rangle$$

5.3. The Heisenberg operators and the KP Hamiltonian. The neutral bilinear operators

$$(55) \quad H_m = \sum_{j \in \mathbb{Z}} : \psi_j \psi_{j+m}^* :, \quad m \in \mathbb{Z}$$

together with the central element 1 form a Heisenberg algebra

$$(56) \quad [H_m, H_n] = m \delta_{m+n,0}, \quad \forall m, n \in \mathbb{Z}$$

and define the KP Hamiltonian

$$(57) \quad H\{t\} = \sum_{m=1}^{\infty} t_m H_m$$

5.4. Boson-fermion correspondence. The character polynomial $\chi_\lambda\{t\}$ can be generated as follows

$$(58) \quad \langle 0 | e^{H\{t\}} \psi_{-b_1}^* \cdots \psi_{-b_d}^* \psi_{a_d} \cdots \psi_{a_1} | 0 \rangle = (-1)^{b_1 + \cdots + b_d} \chi_\lambda\{t\}$$

where a_i is the (length + 1) of the i -th horizontal part, and b_i is the length of the i -th vertical part in the Frobenius decomposition of λ [5], and we assume that $a_d < \cdots < a_1$ and $b_d < \cdots < b_1$, where d is the number of cells on the main diagonal of λ .

5.5. The Bethe scalar product as fermion expectation value. Using the Cauchy-Binet formula to expand the right hand side of Equation (11) in terms of Schur functions, we obtain

$$(59) \quad \det \left(\sum_{k=1}^{N+M-1} h_{k-i}\{x\} \kappa_{k,N-j+1} \right)_{1 \leq i,j \leq N} = \sum_{\lambda \subseteq [(M-1)^N]} c_\lambda s_\lambda\{x\}$$

where

$$(60) \quad c_\lambda = \det \left(\kappa_{(\lambda_{(N-i+1)}+i), (N-j+1)} \right)_{1 \leq i,j \leq N} = \det \left(\kappa_{(\lambda_i+N-i+1), j} \right)_{1 \leq i,j \leq N}$$

Lemma 4. The expansion in terms of Schur functions on the right hand side of Equation (59) can be written as a fermion vacuum expectation value with restricted time variables

$$(61) \quad \sum_{\lambda \subseteq [(M-1)^N]} c_\lambda s_\lambda\{x\} = c_\phi \langle 0 | e^{H\{x\}} e^{X_0\{y\}_\beta} \cdots e^{X_{M-2}\{y\}_\beta} | 0 \rangle$$

where

$$(62) \quad H\{x\} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^N x_i^n H_n$$

and X_j denotes the following sum of fermion bilinears

$$(63) \quad X_j\{y\}_\beta = \sum_{k=1}^N (-)^k d_{[j+1, 1^{(k-1)}]} \psi_{-k}^* \psi_j, \quad j = \{0, \dots, M-2\}$$

with the coefficients $d_\lambda = c_\lambda / c_\phi$, where c_λ is defined in Equation (60).

Proof. The proof of (61) is identical to that of an analogous result for the domain wall partition function in [1]. The differences are that **1.** The coefficients of the expansion of the initial bosonic expression in terms of Schur functions are different, but that does not change the proof, and **2.** The expression that turns out to be a restricted KP τ function is a polynomial of degree $N(M-1)$ rather than $N(N-1)$ in N independent variables $\{x\}$, hence the expansion in terms of Schur functions is indexed by $\lambda \subseteq [(M-1)^N]$ rather than $\lambda \subseteq [(N-1)^N]$.

Using Equations (43) and (61), we obtain the following vacuum expectation expression for the scalar product

$$(64) \quad \langle \{x\} | \{y\}_\beta \rangle = \mathcal{N} \langle 0 | e^{H\{x\}} e^{X_0\{y\}_\beta} \cdots e^{X_{M-2}\{y\}_\beta} | 0 \rangle$$

where the multiplicative factor \mathcal{N} is

$$(65) \quad \mathcal{N} = \frac{c_\phi(q - q^{-1})^N}{\left(\prod_{1 \leq i < j \leq N} (y_j - y_i) \right) \left(\prod_{i \neq j} (y_i q - y_j q^{-1}) \right)}$$

6. A CORRESPONDENCE

Consider the left state $\langle \{x\} |$ on the left hand side of Equation (64). Given that $\{x\}$ are free variables, this state is completely specified by **1.** The length M of the spin chain, **2.** The number N of C operators, **3.** The M inhomogeneity parameters $\{z\}$ and **4.** The crossing parameter γ of the spin chain. But all these data are available in the eigenstate $|\{y\}_\beta\rangle$, hence the scalar product is encoded in and can be recovered from the right state $|\{y\}_\beta\rangle$.

Next, consider the left state $\langle 0 | e^{H\{x\}}$ on the right hand side of Equation (64). Once again, since $\{x\}$ are N free variables, this inner product is completely specified by data that are available in the right state $e^{X_0\{y\}_\beta} \dots e^{X_{M-2}\{y\}_\beta} |0\rangle$. ‘*Peeling off*’ the left states which can be reconstructed from the corresponding right states, we obtain the correspondence

$$(66) \quad |\{\mu\}_\beta\rangle \longrightarrow e^{X_0\{y\}_\beta} \dots e^{X_{M-2}\{y\}_\beta} |0\rangle$$

The left hand side of Equation (66) is an XXZ Bethe eigenstate. The right hand side is a point on Sato’s Grassmannian [2]. The correspondence in Equation (66) is an injective map that assigns to each XXZ eigenstate a point on Sato’s Grassmannian.

7. REMARKS

The fact that quantum integrable models are related to classical integrable differential equations can be traced to the pioneering work [9]. Since then, quite a few results in along these lines have been obtained, particularly on the connection of the quantum Bose gas and the classical nonlinear Schrödinger equation, as reviewed in [3]. Closer to the spirit of the present work is the result of [10] that the quantum XXZ correlation functions at the free fermion point are τ functions of the Ablowitz-Ladik equation². What is new in the present work is the result that the XXZ Bethe scalar product is a KP τ function for all values of the crossing parameter. The Bethe scalar products are basic building blocks of the XXZ correlation functions [11], and we hope that our result will help in the current efforts to compute the latter and their asymptotics.

The correspondence between the Bethe eigenstates and points on Sato’s Grassmannian is reminiscent of Sklyanin’s separation of variables approach to quantum integrable models [12], where every solution to the Bethe equations, $\{y\}_\beta$, labels two objects, **1.** A Bethe eigenstate, and **2.** A function on the projective line with specific monodromy properties. In our correspondence, we obtain a Bethe eigenstate and a point on the Grassmannian. It is also reminiscent of the results of Mukhin *et al.* in the context of the Gaudin limit of the XXX model [13].

Recently, Nekrasov and Shatashvili obtained a correspondence between XXZ Bethe eigenstates and vacuum states of a 3-dimensional super Yang-Mills theory compactified on a circle [14]. Combining the latter correspondence and ours points to a correspondence between a super Yang-Mills theory and points on Sato’s Grassmannian. We hope to explore these issues in future publications.

²We thank N Kitanine for pointing this out to us.

ACKNOWLEDGMENTS

We wish to thank Professors M Jimbo, N Kitanine, V E Korepin, E Mukhin, T Shiota and K Takasaki for useful remarks on this work and on related topics. MW and MZ are supported by Australian Postgraduate Awards and by the Department of Mathematics and Statistics, The University of Melbourne.

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